Recursive and Fractal Structures in Complex Physics

Modern physics increasingly recognizes self‐similar (recursive) patterns in phenomena that seem irregular or divergent. For example, chaotic dynamical systems often exhibit fractal attractors with non‐integer dimensions, and renormalization/group scaling in high‐energy physics is inherently iterative. Researchers have even proposed that space‐time or quantum states may be “fractal” at small scales. These ideas suggest reinterpreting problems via recursion/self‐similarity. Below we summarize how chaos theory, thermodynamics, and quantum physics each display recursive/fractal features and how one can quantify them, drawing on current literature.

Chaos and Nonlinear Dynamics (Fractal Recursion) Figure: Bifurcation diagram of the logistic map as the parameter varies. This classic nonlinear map shows a self‐similar, fractal “cascade” of period-doubling leading to chaos.

Chaotic maps and flows are inherently recursive: each iteration feeds back into the next. A textbook example is the logistic map . Its bifurcation diagram (Fig. above) is self‐similar: zooming into any period‐doubling branch reveals a smaller copy of the whole structure. In fact, it has been proven that the diagram is fractal – each nonchaotic window contains a scaled copy of the map. This means the system “repeats” its structure at different scales. One quantifies this by measuring a fractal (Hausdorff or correlation) dimension of the chaotic attractor. For the logistic map at the onset of chaos (), numerical studies find a Hausdorff dimension ≈0.538 (and a correlation dimension ≈0.50). In other words, the set of states the system visits is a genuine fractal. In practice one can simulate many iterates of the map and estimate the scaling ; this reproduces the known dimension ~0.54, confirming the recursive fractal nature of the attractor.

The same theme appears in other chaotic systems. For instance, the Lorenz attractor (a model of atmospheric convection) is a famous 3D chaotic attractor with fractal geometry. Its shape is the “butterfly” emblem of chaos, but it is not a smooth surface – rather it is an infinite intertwining of strands. It has a non-integer (fractal) dimension that can be computed (roughly ~2.06). In general, strange attractors in both continuous and discrete systems always exhibit self-similar detail. Embedding a Lorenz attractor trajectory (Fig.) shows the recursive folding of trajectories. One can measure its fractal dimension by standard algorithms (e.g. the Grassberger–Procaccia correlation-sum method), confirming that the attractor is not a smooth manifold but a fractal set. These examples illustrate that chaotic maps are perfect “laboratories” of recursion – each iterate reshapes the pattern, yet the overall fractal structure remains.

Another way recursion enters chaos theory is via iterative transformations like renormalization: Feigenbaum’s universality in period-doubling, for example, arises by a recursive rescaling of the map . In summary, chaos theory provides explicit computable signatures of recursion (e.g. fractal dimension, scaling exponents). Thus one can “prove” a system’s recursive structure by measurement: the logistic map’s fractal dimension ≈0.54 or the Lorenz attractor’s ≈2.06 are concrete evidence of self-similarity in the dynamics.

Thermodynamics, Statistical Physics and Fractals

Even in thermodynamics and statistical physics, fractal ideas have emerged. In non-equilibrium or complex systems, entropy and probability distributions often exhibit power laws and multi-scale structure. A prominent example is Tsallis non-extensive statistics: this generalization of Boltzmann–Gibbs entropy was originally motivated by fractals. One can construct a “thermofractal” – a hierarchical system of subsystems that repeats its statistical structure at each level – whose thermodynamics are governed by a fractal dimension. For instance, Deppman (2016) shows that a system with fractal energy distributions obeys Tsallis statistics, and in that model the Hausdorff fractal dimension of the energy support is explicitly related to the entropic index . In his conclusions: “A relation between the fractal dimension and the entropic index is found”, confirming that a recursive phase‐space structure leads to generalized (power-law) thermodynamics.

More generally, complex thermodynamic systems can exhibit fractal phase-space occupation. In turbulent flows or critical phenomena, effective degrees of freedom span many scales, and entropy can be defined on fractal coarse-grainings. For example, in multi-scale models of space (fractal space-time) or in anomalous diffusion, one often introduces fractional dimensions and scale-free (nonlinear) scaling laws. These models remap the usual continuum assumptions onto a self-similar scaffold. Quantitatively, one can derive “scaling exponents” for entropy or correlation functions that replace simple analytic formulas; these exponents are measurable in experiments (e.g. intermittency exponents in turbulent spectra). In principle, one could write down a recursive thermodynamic recursion (partition function at scale in terms of scale ) and solve it numerically. Indeed, fitting experimental particle spectra with Tsallis distributions effectively extracts a fractal index that encodes underlying recursion.

Thus, while classical thermodynamics is built on smooth entropy functions, these fractal-based approaches reframe it in recursive terms. Calculations under this model show that macroscopic quantities (pressure, heat capacity) acquire corrections from the fractal geometry. For example, the energy fluctuations follow rather than an exponential Boltzmann factor. One can numerically simulate such a system of nested subsystems and verify that it reproduces Tsallis statistics and fractal scaling. In short, introducing an explicit recursion (fractal hierarchy) into a thermodynamic model yields new computable predictions (nonlinear scaling laws) that can be tested against data.

Quantum Physics and Fractal Phenomena

Quantum systems also show hints of recursion and fractals. In quantum chaos, where a quantum system has a classically chaotic analog, features of fractality appear in spectra and wavefunctions. For example, Uleysky et al. (2002) studied atoms in a cavity and found that the scattering time vs. initial momentum plots form fractal patterns. They report: “Chaotic wandering of a two-level atom in a quantized Fock field is shown to be fractal. Fractal-like structures, typical for chaotic scattering, are numerically found in the dependence of the exit time on initial momenta”. In other words, even though the underlying evolution is quantum, the outcome displays self-similar intricacy that one can quantify (e.g. by box-counting those time plots).

More recently, quantum entanglement on fractal lattices has been studied. In a 2024 Phys. Rev. Research paper, Zhou and Ye consider fermions on a Sierpinski carpet (a fractal lattice) and find that the entanglement entropy and contour obey new scaling laws. Notably, they observe an emergent “entanglement fractal” pattern in the spatial distribution of entanglement. They write: “we observe the emergence of a novel self-similar and universal pattern termed an ‘entanglement fractal’ in the entanglement contour data”. This is a literal computational example of recursion: the entanglement structure itself repeats fractally across scales, and one can numerically generate it. The authors even provide rules to generate this fractal pattern, showing how quantum correlations iterate in a self-similar way. Such studies demonstrate that when the underlying geometry is recursive, the quantum correlations inherit that recursion in a measurable way (e.g. via scaling of entropy with the fractal dimension).

Finally, in quantum gravity and spacetime models, the notion of fractal dimension is explicitly built-in. Some approaches (like Causal Dynamical Triangulations, non-commutative geometry, or quantum groups) predict that the effective dimension of spacetime changes with scale, often becoming non-integer at the Planck length. For example, Benedetti (2009) showed that certain “quantum spacetime” models have scale‐dependent dimension: at large scales they approach 4D, but at short scales the dimension drops continuously toward 3 or even 2, a signature of fractal behavior. He notes: “quantum groups…have in general a scale-dependent dimension” and in those models the dimensionality becomes non-integral, a typical fractal signature. In practice one calculates a spectral dimension as a function of scale . Numerically (or analytically) one finds at small , e.g. approaching 2 in some models. This is a clear, quantitative recursion: at each scale the “shape” of space changes, yet it repeats (statistically) under scale transformations. These predictions can be checked by lattice quantum gravity simulations or diffusion algorithms, yielding numeric evidence of fractal spacetime geometry.

Summary of Recursive Modeling and Calculations

In all these cases—chaos, thermodynamics, quantum theory—we see recursion embodied by self-similarity or iterative scaling. Mathematically, one can often remap the problem: instead of a single continuum description, consider a hierarchy of similar levels or a multi-scale transform. Then one derives new equations (sometimes integral or difference equations) that link one scale to the next. Solving these numerically or analytically yields novel relations. For instance, iterating the logistic map and computing the set of points (orbit) gives a bifurcation plot (above) and a fractal attractor with dimension ~0.54. In thermodynamics, iterating a hierarchical “thermofractal” yields Tsallis-like energy distributions. In quantum models, iterating many scales of geometry changes the effective entropy scaling or space-time dimension.

Concrete calculation examples: We can simulate these systems to measure their recursion. For the logistic map at , one iterates points and bins them to estimate the attractor’s fractal dimension – this recovers in agreement with the literature. For a thermofractal model, one could write a computer routine that generates nested energy partitions according to a scaling rule and then computes the resulting entropy (which will follow a -exponential form). In quantum fractal systems, one can numerically diagonalize a Hamiltonian on a fractal lattice and compute entanglement entropy or correlation functions; the results will exhibit the predicted “entanglement fractal” pattern. Each of these is a “proof by calculation” of the underlying recursion: the measurable outcomes (fractal dimension, entropy scaling, etc.) verify the self-similar model.

In conclusion, applying a recursive/fractal framework can indeed recast and potentially simplify aspects of these fields. It provides quantitative handles (dimensions, scaling exponents, non-integer statistics) for phenomena where traditional smooth mathematics struggles (e.g. turbulence, quantum gravity divergences, chaotic unpredictability). While such approaches are still under development, they have already yielded testable predictions and numerical models. For example, chaos theory shows that a simple iterative map has an exactly computable fractal structure; thermodynamic fractal models naturally produce Tsallis entropy laws; and quantum models on fractal spaces reveal new entropy scaling. Each case demonstrates that “proving recursion” – i.e. demonstrating self-similarity through measurable quantities – is feasible with current theory and computation.

Sources: Key results and definitions are drawn from the literature on chaos and fractals, on fractal thermodynamics, and on quantum fractals. These works illustrate how recursive/self-similar structure emerges and can be quantified in advanced physical models.